

Leading infrared logarithms for σ -model with fields on arbitrary Riemann manifold*

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Abstract

We derive non-linear recursion equation for the leading infrared logarithms (LL) in four dimensional σ -model with fields on an arbitrary Riemann manifold. The derived equation allows one to compute leading infrared logarithms to essentially unlimited loop order in terms of geometric characteristics of the Riemann manifold.

We reduce the solution of the $SU(\infty)$ principal chiral field in arbitrary number of dimensions in the LL approximation to the solution of very simple recursive equation. This result paves a way to the solution of the model in arbitrary number of dimensions at $N \rightarrow \infty$.

1 Introduction

Effective Field Theories (EFTs) are *non-renormalizable* field theories, which allow the investigation of the infrared (low-energy) behaviour of various physical systems (see different examples in Refs. [1, 2]). The standard tool for the studies of the asymptotic behaviour of *renormalizable* field theories is the method of renormalization group equations (RGEs). In the case of EFTs the method of RGEs must be modified as the number of counterterms increases rapidly with the loop order. For subtleties of the renormalization in quantum field theory see an excellent book by Alexandre Nikolaevich Vasiliev [1].

A possibility of the systematic construction of RGEs for non-renormalizable quantum field theories was demonstrated in Ref. [3]. In particular, it was shown that the series of the leading logarithms (LLs) can be obtained by

*Devoted to memory of Alexandre Nikolaevich Vasiliev.

calculation of one loop diagrams. However, the solution of the RGEs derived in Ref. [3] requires the calculation of non-trivial one-loop diagrams. The number of the diagrams increases rapidly with the loop order. Therefore, the implementation of this method in practice is not an easy task. The method of Ref. [3] has been applied in Ref. [4] for the calculation of LLs up to five-loops for the decay constant and pion mass, and up to four-loops for meson-meson scattering in the massive $O(N+1)/O(N)$ sigma model. In Ref. [5] the authors, using dispersive methods, calculated the three-loop LLs to $\pi\pi$ scattering in massless Chiral Perturbation Theory (ChPT).

Recently, a completely different method for the calculation of LLs in a wide class of non-renormalizable *massless* field theories was developed in Refs. [6, 7, 8]. The non-linear recursion equations derived in Refs. [6, 7, 8] allow one to obtain the LLs contributions to practically unlimited loop order without performing non-trivial loop calculations at each loop order.

In the present paper we compute the leading infrared logs contribution for the four dimensional σ -model with fields on arbitrary Riemann manifold. One of our main aim here is to obtain the interpretation of LLs in terms of geometric characteristics of the Riemann manifold. In particular, this approach allows us to reduce the solution of the $SU(\infty)$ principal chiral field in arbitrary number of dimensions in the LL approximation to the solution of very simple recursive equation. This result paves a way to the solution of the model in arbitrary number of dimensions at $N \rightarrow \infty$.

The paper is organized as follows. The most general main results of this paper are given in Eqs. (6,7) of Section 2. We also give in Section 2, it seems for the first time, 2-loop results for general σ -model, see Eq. (12). The way how the general method of Section 2 works is demonstrated on the “routine example” of N dimensional sphere as a target space in Section 3. The most interesting and non-trivial results for $SU(\infty)$ principal chiral field are given by Eqs. (23,24) of Section 4.

2 The σ -model with fields on arbitrary Riemann manifold

We consider the most general σ -model with the fields on an arbitrary Riemann manifold. For two space-time dimensions the corresponding model is renormalizable [9] and LL infrared asymptotic is determined by the one-

loop RGE, which is just the Ricci flow equation for the metric. In higher space-time dimensions the calculation of the LLs is highly non-trivial task, to our best knowledge only one loop LLs in four dimensions were computed for a general σ -model [10, 11]. Here we show how to compute the LLs to an arbitrary loop order for a general σ model in four space-time dimensions[†].

The action is given by the following expression:

$$S = \int d^4x \frac{1}{2} g_{ab}(\phi) \partial_\mu \phi^a \partial_\mu \phi^b, \quad (1)$$

where $g_{ab}(\phi)$ is a metric on a Riemann manifold. Without loss of generality we consider here a compact Riemann manifold with positive signature of the metric. Using the freedom of the coordinates (ϕ^a) choice on the Riemann manifold we fix the metric such that $g_{ab}(0) = \delta_{ab}$ and $g_{ab}(\phi)\phi^b = \delta_{ab}\phi^b$. The latter condition corresponds to the choice of the normal coordinates on the Riemann manifold [12]. In these coordinates the geodesics in the vicinity of $\phi^a = 0$ are simple straight lines and the metric has the following expansion around $\phi^a = 0$:

$$g_{ab}(\phi) = \delta_{ab} - \frac{1}{3} \overset{\circ}{R}_{acbd} \phi^c \phi^d + O(\phi^3), \quad (2)$$

where $\overset{\circ}{R}_{acbd}$ is the Riemann curvature tensor at $\phi^a = 0$. Substituting the expansion (2) in the action (1) we can easily compute the tree level scattering amplitude of $\phi^a + \phi^b \rightarrow \phi^c + \phi^d$ with the result:

$$A_{abcd}^{\text{tree}}(s, \cos \theta) = \frac{s}{2} \left(- \left[\overset{\circ}{R}_{acbd} + \overset{\circ}{R}_{adbc} \right] P_0(\cos \theta) + \overset{\circ}{R}_{abcd} P_1(\cos \theta) \right). \quad (3)$$

Here s is the Mandelstam variable, θ is the centre of mass scattering angle and P_l are Legendre polynomials. This expression provides the leading infrared asymptotic of the scattering amplitude. The corrections to this result arise from the loop contributions. Simple power counting arguments show that the $(n-1)$ -loop contribution has the following form of the LLs contribution $s[s \ln(\mu^2/s)]^{n-1}$. For the l 's partial wave amplitude defined as:

$$t_{abcd}^l(s) = \frac{1}{64\pi} \int_0^\pi d\theta \sin \theta A_{abcd}(s, \cos \theta) P_l(\cos \theta), \quad (4)$$

[†]The generalization to $D \geq 4$ is simple and can be done using method of Ref. [8].

we can write the general form of the LL expansion:

$$t_{abcd}^l(s) = \frac{\pi}{2} \frac{1}{2l+1} \sum_{n=1}^{\infty} (\omega_{nl})_{abcd} \left(\frac{s}{16\pi^2} \right)^n \ln^{n-1} \left(\frac{\mu^2}{s} \right) + \mathcal{O}(\text{NLL}). \quad (5)$$

Here $(\omega_{nl})_{abcd}$ are the LLs coefficients which depend only on the geometric characteristics of the target space. These coefficients are non-zero only for $l \leq n$, where $(n-1)$ is the number of loops. The tree level amplitude (3) corresponds to $n=1$, in this case only $l=0$ and $l=1$ coefficients are non-zero:

$$\begin{aligned} (\omega_{10})_{abcd} &= -\frac{1}{2} \left(\overset{o}{R}_{acbd} + \overset{o}{R}_{adbc} \right), \\ (\omega_{11})_{abcd} &= \frac{1}{2} \overset{o}{R}_{abcd}. \end{aligned} \quad (6)$$

Now we can use the general method of Ref. [8] to obtain the recursion equation which allows us to express the higher loop LL coefficients in terms of the tree level ones (6). The method of Ref. [8] is based on the general principles of a quantum field theory – analyticity, unitarity and crossing. We refer the reader to [8] for the description of the general method, here we present the final result for the recursion equation:

$$\begin{aligned} (\omega_{nl})_{abcd} &= \frac{1}{2(n-1)} \left[\sum_{i=1}^{n-1} \frac{(\omega_{il})_{ab\alpha\beta} (\omega_{n-i,l})^{\beta\alpha}_{cd}}{2l+1} \right. \\ &\quad \left. + \sum_{i=1}^{n-1} \sum_{l'=0}^{n-1} \frac{(\omega_{il'})_{ad\alpha\beta} (\omega_{n-i,l'})^{\beta\alpha}_{cb} \Omega_n^{l'l} + (-1)^{l+l'} (\omega_{il'})_{ac\alpha\beta} (\omega_{n-i,l'})^{\beta\alpha}_{bd} \Omega_n^{l'l}}{2l'+1} \right]. \end{aligned} \quad (7)$$

Here $\Omega_n^{l'l}$ is the crossing matrix in the angular momentum space, it has the following form:

$$\Omega_n^{l'l} = \frac{2l+1}{2^{n+1}} \int_{-1}^1 dz P_{l'} \left(\frac{z+3}{z-1} \right) P_l(z) (z-1)^n. \quad (8)$$

Eq. (7) expresses the higher loop LL coefficients in terms of the lower loop. The starting point for the recursion is given by the tree level expressions (6).

One can easily check that the solution of the equation (7) satisfies the following symmetries:

$$(\omega_{nl})_{abcd} = (-1)^l (\omega_{nl})_{bacd} = (-1)^l (\omega_{nl})_{abdc} = (\omega_{nl})_{cdab}. \quad (9)$$

These symmetries are actually the consequence of the Bose symmetry and the parity conservation. Additionally Eq. (7) posses the following symmetries:

$$\left(\sum_{l'=0}^n \omega_{nl'} \Omega_n^{l'l} \right)_{abcd} = (\omega_{nl})_{adcb}, \quad \left(\sum_{l'=0}^n \omega_{nl'} (-1)^{l+l'} \Omega_n^{l'l} \right)_{abcd} = (\omega_{nl})_{acbd}, \quad (10)$$

which might be very useful for possible exact solution of the recursion equation (7).

We did not find yet the analytic solution of the equation (7), therefore we give here expressions for the LL coefficients to the 2-loop order. The 1-loop LL coefficients in terms of geometric characteristics have the following form:

$$\begin{aligned} (\omega_{20})_{abcd} &= \frac{1}{4} \left[R_{a\beta_1 b\beta_2} R_c^{\beta_1 \beta_2} + R_{a\beta_1 b\beta_2} R_c^{\beta_2 \beta_1} \right] \\ &+ \frac{1}{6} \left[R_{a\beta_1 c\beta_2} R_b^{\beta_1 \beta_2} + R_{a\beta_1 d\beta_2} R_b^{\beta_1 \beta_2} \right] - \frac{5}{72} \left[R_{ac\beta_1\beta_2} R^{\beta_1\beta_2}_{bd} + R_{ad\beta_1\beta_2} R^{\beta_1\beta_2}_{bc} \right], \\ (\omega_{21})_{abcd} &= -\frac{1}{24} R_{ab\beta_1\beta_2} R^{\beta_1\beta_2}_{cd} \\ &+ \frac{1}{4} \left[R_{a\beta_1 c\beta_2} R_b^{\beta_1 \beta_2} - R_{a\beta_1 d\beta_2} R_b^{\beta_1 \beta_2} \right] - \frac{1}{12} \left[R_{ac\beta_1\beta_2} R^{\beta_1\beta_2}_{bd} - R_{ad\beta_1\beta_2} R^{\beta_1\beta_2}_{bc} \right], \\ (\omega_{22})_{abcd} &= \frac{1}{12} \left[R_{a\beta_1 c\beta_2} R_b^{\beta_1 \beta_2} + R_{a\beta_1 d\beta_2} R_b^{\beta_1 \beta_2} \right] \\ &- \frac{1}{72} \left[R_{ac\beta_1\beta_2} R^{\beta_1\beta_2}_{bd} + R_{ad\beta_1\beta_2} R^{\beta_1\beta_2}_{bc} \right]. \end{aligned} \quad (11)$$

Here and below all Riemann tensors are at $\phi^a = 0$. The 2-loop result is the following:

$$\begin{aligned} (\omega_{30})_{abcd} &= \chi_{abcd} - \frac{1}{2} \psi_{abcd} + \frac{1}{3} \psi_{adbc} - \frac{1}{4} \chi_{adbc}, \\ (\omega_{31})_{abcd} &= \frac{1}{2} \psi_{abcd} - \frac{1}{2} \psi_{adbc} - \frac{9}{20} \chi_{adbc}, \\ (\omega_{32})_{abcd} &= \frac{1}{6} \psi_{adbc} - \frac{1}{4} \chi_{adbc}, \\ (\omega_{33})_{abcd} &= \frac{1}{20} \chi_{adbc}, \end{aligned} \quad (12)$$

where

$$\begin{aligned}
\chi_{abcd} = & \left[R_{a\beta_1 b\beta_2} \left(-\frac{1}{72} R_{c\beta_3}^{\beta_1} R_d^{\beta_3\beta_2\beta_4} - \frac{7}{144} R_{c\beta_3}^{\beta_1} R_d^{\beta_4\beta_2\beta_3} \right. \right. \\
& \left. \left. - \frac{1}{48} R_{c\beta_3}^{\beta_2} R_d^{\beta_4\beta_1\beta_3} \right) + (a \rightarrow b \rightarrow d \rightarrow c \odot) \right. \\
& + \frac{1}{144} R_{a\beta_1 b\beta_2} (R_{c\beta_3 d\beta_4} - R_{c\beta_4 d\beta_3}) R^{\beta_1\beta_2\beta_3\beta_4} \\
& \left. - \frac{1}{8} R_{a\beta_1 b\beta_2} (R_{c\beta_3 d\beta_4} + R_{c\beta_4 d\beta_3}) R^{\beta_1\beta_3\beta_2\beta_4} \right] \\
& + (c \leftrightarrow b),
\end{aligned}$$

$$\begin{aligned}
\psi_{abcd} = & \left[R_{ab\beta_1\beta_2} \left(-\frac{1}{72} R_{c\beta_3}^{\beta_1} R_d^{\beta_3\beta_2\beta_4} - \frac{1}{36} R_{c\beta_3}^{\beta_1} R_d^{\beta_4\beta_2\beta_3} \right) \right. \\
& \left. + \begin{pmatrix} a \leftrightarrow c \\ b \leftrightarrow d \end{pmatrix} - (b \leftrightarrow c) - (a \rightarrow b \rightarrow d \rightarrow c \odot) \right] \\
& + \left[R_{b\beta_1 d\beta_2} \left(\frac{1}{36} R_{a\beta_3}^{\beta_2} R_c^{\beta_3\beta_1\beta_4} + \frac{1}{72} R_{a\beta_3}^{\beta_1} R_c^{\beta_3\beta_2\beta_4} \right. \right. \\
& \left. \left. + \frac{17}{144} R_{a\beta_3}^{\beta_2} R_c^{\beta_4\beta_2\beta_3} + \frac{13}{144} R_{a\beta_3}^{\beta_1} R_c^{\beta_4\beta_2\beta_3} \right) \right. \\
& \left. + \begin{pmatrix} a \leftrightarrow b \\ c \leftrightarrow d \end{pmatrix} \right] + \frac{3}{16} R_{a\beta_1 c\beta_2} R_{b\beta_3 d\beta_4} R^{\beta_1\beta_3\beta_2\beta_4} \\
& + \left(\frac{1}{144} R_{ab\beta_1\beta_2} R_{cd\beta_3\beta_4} - \frac{1}{72} R_{a\beta_1 c\beta_2} R_{b\beta_3 d\beta_4} \right) R^{\beta_1\beta_2\beta_3\beta_4}
\end{aligned}$$

The expressions for higher loops can be easily obtained by further iterations of Eq. (7). We do not give them, as the corresponding expressions are rather busy. Instead, we discuss below the particular cases of the Riemann manifolds: the N -dimensional sphere S^N and the group manifold corresponding to a simple Lie group G . The latter case is the principal chiral field σ -model that has much in common with Yang-Mills theory.

3 The σ -model with fields on a sphere S^N

The metric on the N -dimensional sphere S^N in the normal coordinates has the following form:

$$g_{ab}(\phi) = F^2 \left(\delta_{ab} \frac{\sin^2 \left(\frac{|\phi|}{F} \right)}{|\phi|^2} + \frac{\phi_a \phi_b}{|\phi|^2} \left[\frac{1}{F^2} - \frac{\sin^2 \left(\frac{|\phi|}{F} \right)}{|\phi|^2} \right] \right), \quad (13)$$

where $|\phi| \equiv \sqrt{\phi_a \phi^a}$ and F is the radius of S^N . For the theory (1) in four space-time dimensions the constant F has the dimension of mass, and for S^3 corresponds to the pion decay constant of the effective chiral Lagrangian of two flavour QCD. The Riemann and Ricci tensors, and scalar curvature for S^N have the following form:

$$R_{abcd} = \frac{1}{F^2} (g_{ac}g_{bd} - g_{ad}g_{bc}), \quad R_{ab} = \frac{N-1}{F^2} g_{ab}, \quad R = \frac{N(N-1)}{F^2}. \quad (14)$$

Substituting these expressions to the general solutions (11,12) one obtains:

$$\begin{aligned} (\omega_{20})_{abcd} &= \frac{1}{18F^4} [(3N-1)(\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}) + (9N-17)\delta_{ab}\delta_{cd}], \quad (15) \\ (\omega_{21})_{abcd} &= \frac{1}{4F^4} (N-3)(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}), \\ (\omega_{22})_{abcd} &= \frac{1}{36F^4} [(3N-5)(\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}) + 4\delta_{ab}\delta_{cd}], \end{aligned}$$

for 1-loop LL coefficients. The 2-loop coefficients have the following form:

$$\begin{aligned} (\omega_{3l})_{abcd} &= \frac{1}{288F^6} [a_{3l} (\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}) + b_{3l} \delta_{ab}\delta_{cd}], \quad \text{for } l = 0, 2 \quad (16) \\ (\omega_{3l})_{abcd} &= \frac{1}{1440F^6} c_{3l} (\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}), \quad \text{for } l = 1, 3 \end{aligned}$$

with

$$\begin{aligned} a_{30} &= -59 + 27N - 18N^2, \quad b_{30} = 86 - 158N + 72N^2, \quad (17) \\ a_{32} &= -39 + 47N - 18N^2, \quad b_{32} = 10 + 10N, \\ c_{31} &= 441 - 333N + 162N^2, \quad c_{33} = 49 - 37N + 18N^2. \end{aligned}$$

In the above results for the first two loops one can note a systematic pattern for the leading large N coefficients. Indeed, one can easily derive[‡] the large

[‡]One has to use the isometries of S^N manifold, see next Section.

N solution of Eq. (7) for the case of S^N [6, 8]. The corresponding solution has the form:

$$\begin{aligned}
(\omega_{nl})_{abcd}^{\text{Large } N} &= \frac{1}{F^2} \left(-\frac{N}{2F^2} \right)^{n-1} \left[\frac{(2l+1)n!}{(n+l+1)!} (\delta_{ad}\delta_{bc} + (-1)^l \delta_{ac}\delta_{bd}) \right. \\
&\quad \left. + (-1)^{n-1} \delta_{l0} \delta_{ab}\delta_{cd} \right]. \tag{18}
\end{aligned}$$

We see that the σ -model with fields on S^N can be solved in the large N -limit for arbitrary number of space-time dimensions. It is well known result. The record calculations of the $1/N$ corrections were performed by A.N. Vasiliev et al. [13] by very elegant method of the conformal bootstrap. From the large N solution (18) we see that the amplitude, after summation of LL's, has a pole for $l = 0$ and $O(N)$ singlet channel. This pole corresponds to the contribution of the auxiliary scalar field which one introduces to solve the sigma model on S^N in the large N limit.

We note that all results obtained in this section for S^N are valid also for any simple connected conformally flat Riemann manifold.

4 Principal chiral field, $SU(\infty)$ case especially

An important case is the σ -model with the target space $\frac{G \times G}{G}$, where G is a simple Lie group. This is the model of principal chiral field. The metric in normal coordinates on $\frac{G \times G}{G}$ has the form:

$$g_{ab}(\phi) = \int_0^1 d\alpha (1 - \alpha) \text{tr} \left(\exp \left(i\alpha \frac{t^c \phi^c}{F} \right) t^a \exp \left(-i\alpha \frac{t^c \phi^c}{F} \right) t^b \right), \tag{19}$$

where t^a are the generators of the group G in the fundamental representation which are normalized by $\text{tr}(t^a t^b) = 2\delta^{ab}$, F is the parameter of mass dimension one. For $G = SU(N)$ this parameter corresponds to Nambu-Goldstone boson decay constant in the chiral Lagrangian for QCD with N massless quark flavours. For what follows we consider the case of $G = SU(N)$. The $SU(N)$ principal chiral field model in many respects is similar to the Yang-Mills theory, for example, its large N limit corresponds to the summation of the planar diagrams. Despite many hopes [14] this model is not solved in the large N limit for arbitrary number of space-time dimensions. We shall discuss this limit in LL approximation below.

The Riemann and Ricci tensors, and the scalar curvature for $SU(N) \times SU(N)/SU(N)$ manifold have the following form:

$$R_{abcd} = \frac{1}{8F^2} \text{tr}([t^a, t^b][t^c, t^d]) + O(\phi^2), \quad R_{ab} = \frac{N}{F^2} g_{ab}, \quad R = \frac{N(N^2 - 1)}{F^2}. \quad (20)$$

In principle, one can use these expressions for the our main equations (7,6) to obtain LL coefficients. However, the results, especially for high loop orders, are rather cumbersome. Therefore, at this point it is wise to use the isometries of the $SU(N) \times SU(N)/SU(N)$ manifold. For the case of the amplitude (5) the isometries allow us to decompose the amplitude in the projectors onto the irreducible representations of the $SU(N)$ group:

$$t_{abcd}^l(s) = \sum_{R=1}^7 P_{abcd}^R t_R^l(s), \quad (21)$$

where P_{abcd}^R are projectors onto the irreducible representations of the $SU(N)$ group which arise in the product of $\text{Adj} \times \text{Adj}$. Generically, these are seven representations with the dimensions [§]:

$$d_R = \left(1, N^2 - 1, \frac{N^2(N+1)(N-3)}{4}, \frac{N^2(N-1)(N+3)}{4}, \right. \\ \left. N^2 - 1, \frac{(N^2 - 4)(N^2 - 1)}{4}, \frac{(N^2 - 4)(N^2 - 1)}{4} \right). \quad (22)$$

The derivation and the explicit expressions for the projectors P_{abcd}^R can be found in [15]. We note that the representations with numbers $R = 2$ and $R = 5$ correspond to the symmetric and the antisymmetric adjoint representations of the $SU(N)$ group. In Ref. [15] it was shown that in the large N limit only these two adjoint representations survive, the contribution of other representation is the $1/N$ correction. That is very interesting observation, because it allows one to reduce considerably the number of geometric objects in the large N limit.

The solution for the LL coefficients to *arbitrary loop order* at $N \rightarrow \infty$ for the $SU(N)$ principal chiral field has the following form:

$$(\omega_{nl})_{abcd}^{\text{Large } N} = \frac{\rho_{nl}}{16F^2} \left(\frac{N}{2F^2} \right)^{n-1} \text{tr}([t^a, t^b][t^c, t^d]), \quad \text{for odd } l, \quad (23)$$

[§]For the reader experienced in $SU(3)$ these are $d_R = (1, 8_S, -, 27, 8_A, 10, \overline{10})$.

where the coefficients ρ_{nl} satisfy simple recursion equation:

$$\rho_{nl} = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \sum_{l'=0}^n \frac{\rho_{il'} \rho_{n-i,l'}}{2l'+1} \left(\delta^{l'l} + \Omega_n^{l'l} \right), \quad (24)$$

with the initial conditions $\rho_{10} = \rho_{11} = 1$. The expression for even l is rather complicated and we do not have place to show it. It can be easily obtained from Eq. (23) with help of relations (10). In any case, the amplitude for even l is expressed in terms of the same coefficients ρ_{nl} (24).

We reduced the solution of the $SU(\infty)$ principal chiral field in arbitrary number of dimensions in the LL approximation to the solution of very simple recursive equation (24). It is remarkable equation, it possesses rich symmetries. Unfortunately we did not find yet its analytical solution[¶], however it can be solved numerically to practically unlimited loop order. Studies of this equation we shall present elsewhere.

5 Conclusions and outlook

We derived the non-linear recursion relation (7) for the LL coefficients in the four dimensional sigma model with fields on arbitrary Riemann manifold. Being supplemented by the initial conditions (6) this equation allows one to obtain the LL coefficients in terms of the geometric characteristics of the Riemann manifold. We calculated explicitly two loop LLs for arbitrary σ -model. We can speculate that it might be possible to obtain Eq. (7) as a some kind of equations of motion for a non-local object in the Riemann manifold. If one would manage this, the LL coefficient could be obtained by the expansion of the object in its non-locality.

Eq. (7) being applied to 4D $SU(N)$ principal chiral field allowed us to reduce the solution of this theory at $N \rightarrow \infty$ and LL approximation to the solution of simple and nice equation (24). We hope that this equation can be solved analytically with help of rich symmetries it possesses. Such solution can provide a clue for the spectrum of $SU(\infty)$ principal chiral field in arbitrary number of space-time dimensions.

[¶]As a side remark we note that Eq. (24) without the term $\sim \Omega_n^{l'l}$ has very simple solution $\rho_{nl} = \frac{1}{2^{n-1}} \delta_{l0} + \frac{1}{6^{n-1}} \delta_{l1}$.

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